

On Continuous Ambiguities in Model-Independent Partial Wave Analysis - I.

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Abstract

A problem of amplitude reconstruction in terms of the given angular distribution is considered. Solution of this problem is not unique. A class of amplitudes, correspondent to one and the same angular distribution, forms a region in projection onto a finite set of spherical harmonics. An explicit parametrization of a boundary of the region is obtained. A shape of the region of ambiguities is studied in particular example. A scheme of partial-wave analysis, which describes all solutions in the limits of the region, is proposed.

I. Let a quantum system, described by a wave function Ψ , decay into 2 spinless particles. We are studying angular distribution of decay products in the rest frame of initial state

$$I(\vec{n}) = |\Psi(\vec{n})|^2$$

and are going to obtain the wave function $\Psi(\vec{n})$ in the form of expansion by spherical harmonics

$$\Psi(\vec{n}) = \sum_{lm} c_{lm} Y_{lm}(\vec{n}) .$$

The solution of the problem:

$$|\Psi(\vec{n})| = \sqrt{I(\vec{n})}, \quad \Psi(\vec{n}) = \sqrt{I(\vec{n})} e^{i\varphi(\vec{n})}, \quad (1)$$

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$\varphi(\vec{n})$ is an arbitrary function of angles,

$$c_{lm} = \int d^2n Y_{lm}^*(\vec{n}) \sqrt{I(\vec{n})} e^{i\varphi(\vec{n})}. \quad (2)$$

Substituting all possible functions $\varphi(\vec{n})$ into this formula, one obtains the sets of coefficients c_{lm} , correspondent to the same distribution $I(\vec{n})$. In this the values of the coefficients span in the space $\{c_{lm}\}$ some region Ω . Let us obtain a boundary of this region.

We denote $(lm) = \alpha$. Let us find the variation of coefficients when $\varphi(\vec{n})$ changes

$$\delta c_\alpha = \int d^2n Y_\alpha^*(\vec{n}) \sqrt{I(\vec{n})} e^{i\varphi(\vec{n})} i\delta\varphi(\vec{n}).$$

On the boundary all δc_α are orthogonal to a single vector N_α – normal to the boundary surface (fig.1).

We introduce a scalar product in the space of coefficients in the following way:

$$(a, b) = \frac{1}{2} \sum_\alpha (a_\alpha^* b_\alpha + a_\alpha b_\alpha^*) = \sum_\alpha \text{Re } a_\alpha \text{Re } b_\alpha + \sum_\alpha \text{Im } a_\alpha \text{Im } b_\alpha. \quad (3)$$

$$(N, \delta c) = \frac{i}{2} \sum_\alpha \int d^2n \sqrt{I} (N_\alpha^* Y_\alpha^* e^{i\varphi} - N_\alpha Y_\alpha e^{-i\varphi}) \delta\varphi(\vec{n}) = 0 \quad \forall \delta\varphi(\vec{n})$$

$$\Rightarrow \sum_\alpha N_\alpha^* Y_\alpha^* e^{i\varphi} - \sum_\alpha N_\alpha Y_\alpha e^{-i\varphi} = 0 \Rightarrow e^{2i\varphi(\vec{n})} = \frac{\sum_\alpha N_\alpha Y_\alpha(\vec{n})}{\sum_\alpha N_\alpha^* Y_\alpha^*(\vec{n})}$$

$$e^{i\varphi(\vec{n})} = \pm \sqrt{\frac{\sum_\alpha N_\alpha Y_\alpha(\vec{n})}{\sum_\alpha N_\alpha^* Y_\alpha^*(\vec{n})}} \quad \text{or}$$

$$\varphi(\vec{n}) = \arg \sum_\alpha N_\alpha Y_\alpha(\vec{n}) \quad (+\pi, \text{ for bottom sign}). \quad (4)$$

This function $\varphi(\vec{n})$ is correspondent to the point, lying on the boundary of the region Ω

$$c_\beta(N) = \pm \int d^2n \sqrt{I(\vec{n})} Y_\beta^*(\vec{n}) e^{i \arg \sum N_\alpha Y_\alpha(\vec{n})}. \quad (5)$$

The sign \pm in this formula is excessive. The region Ω is symmetrical in reflections about the origin, because the angular distribution does not change in the replacement $\Psi \rightarrow -\Psi$ ($c \rightarrow -c$). For the points on the boundary this

replacement is equivalent to the transformation $N \rightarrow -N$. In fact, sign ambiguity in (5) corresponds to the replacement of outer normals to inner ones.

Phase (4) does not change in multiplication of all N_α by positive number. One can choose the unit normal N_α : $(N, N) = \sum_\alpha |N_\alpha|^2 = 1$.

Let the unit normal N be given on a surface Γ . Thereby a mapping of the surface into unit sphere S is specified:

$$\Gamma \xrightarrow{N} S, \quad N = N(c), \quad c \in \Gamma, \quad N \in S.$$

This mapping is called Gaussian mapping [1]. It possesses the property equivalent to the definition:

$$(N, dc) = 0.$$

Formula (5) specifies a parametrization of the surface $\Gamma = \partial\Omega$, the parameter is unit normal N . Thus Γ is defined by the inverse Gaussian mapping

$$S \xrightarrow{c} \Gamma, \quad c = c(N), \quad N \in S, \quad c \in \Gamma.$$

In order to obtain *a projection* of the region Ω in some subspace $\{c_{lm}, l = 0..L_{\max}\}$, one should restrict N_α in formula (5) into this subspace. Values c_β obtained specify a boundary of the projection of the region (fig.2).

In order to obtain *a slice* of the region Ω by specified hyperplane $\{c_{lm} = c_{lm}^{(0)} - \text{fixed numbers}, l > L_{\max}\}$, one should solve a system of equations on N_α

$$\int d^2n \sqrt{I} Y_\beta^* e^{i \arg \sum N_\alpha Y_\alpha} = c_\beta^{(0)}.$$

This problem is more difficult than preceding one.

Example. Let $I = \frac{1}{4\pi}$. What initial states (except the obvious S-wave $\Psi = \frac{1}{\sqrt{4\pi}}$) are able to give this distribution?

The answer: $\Psi = \frac{1}{\sqrt{4\pi}} e^{i\varphi(\vec{n})}$. The boundary of the region Ω is given by the formula

$$c_\beta = \frac{1}{\sqrt{4\pi}} \int d^2n Y_\beta^* e^{i \arg \sum N_\alpha Y_\alpha}. \quad (6)$$

Now the problem consists in clear representation of this region.

1) Projection of Ω in finite dimensional space $\{c_{lm}, l = 0..L_{\max}\}$ lies inside unit sphere

$$\sum_0^{L_{\max}} c_{\alpha}^* c_{\alpha} \leq \sum_0^{\infty} c_{\alpha}^* c_{\alpha} = \int d^2 n |\Psi|^2 = 1.$$

2) Projection of Ω on each wave LM covers a circle inside unit circle on Argand plot:

$$\text{when } N_{lm} = \begin{cases} e^{i\phi}, & l = L, m = M \\ 0, & \text{otherwise} \end{cases} \quad \arg \sum N_{\alpha} Y_{\alpha} = \phi + \arg Y_{LM},$$

$$c_{LM} = \frac{1}{\sqrt{4\pi}} e^{i\phi} \int d^2 n |Y_{LM}| e^{-i \arg Y_{LM}} \cdot e^{i \arg Y_{LM}} = \frac{1}{\sqrt{4\pi}} e^{i\phi} \int d^2 n |Y_{LM}| =$$

$(L, M) :$	$(0, 0)$	$(1, 0)$	$(1, 1)$	$(2, 0)$	$(2, 1)$	$(2, 2)$
$= \frac{1}{2} e^{i\phi} \int_{-1}^1 d \cos \theta \cdot$	1	$\sqrt{3} \cos \theta $	$\sqrt{\frac{3}{2}} \sin \theta$	$\frac{\sqrt{5}}{2} 3 \cos^2 \theta - 1 $	$\sqrt{\frac{15}{2}} \sin \theta \cos \theta $	$\sqrt{\frac{15}{8}} \sin^2 \theta$
$= e^{i\phi}.$	1;	$\frac{\sqrt{3}}{2},$	$\frac{\pi}{4} \sqrt{\frac{3}{2}};$	$\frac{2}{3} \sqrt{\frac{5}{3}},$	$\sqrt{\frac{5}{6}},$	$\sqrt{\frac{5}{6}}$
	1.;	0.962,	0.866;	0.861,	0.913,	0.913

$$(c_L - M = c_{LM})$$

3) Projection on S,P₀-waves (a space $\{c_0, c_{10}\}$).

When rotating phases of all N_{α} by ϕ_0 , the phases c_{α} also rotate by ϕ_0 :

$$N_{\alpha} \rightarrow N_{\alpha} e^{i\phi_0} \Rightarrow c_{\alpha} \rightarrow c_{\alpha} e^{i\phi_0}$$

In multiplication of N_{α} by a positive number the c_{α} do not change.

Let $N_0 = 1$. We will measure the phase of P-wave with respect to the phase of S-wave

$$c_{10} \rightarrow c_{10} e^{-i \arg c_0}, c_0 \rightarrow |c_0|. \quad (7)$$

Let us denote $\sqrt{3} N_{10} = \rho e^{i\phi}$

$$e^{i \arg(1 + \rho e^{i\phi} \cos \theta)} = \frac{1 + \rho \cos \theta e^{i\phi}}{\sqrt{1 + 2\rho \cos \theta \cos \phi + \rho^2 \cos^2 \theta}}.$$

From (6)

$$c_0 = \frac{1}{4\pi} \int_0^{2\pi} d\varphi \cdot \int_{-\rho}^{\rho} \frac{dx}{\rho} \frac{1 + x e^{i\phi}}{\sqrt{1 + 2x \cos \phi + x^2}} \quad x = \rho \cos \theta$$

$$\begin{aligned}
&= \frac{\sin \phi}{2\rho} \int_{(-\rho+\cos \phi)/\sin \phi}^{(\rho+\cos \phi)/\sin \phi} du \frac{1 - e^{i\phi} \cos \phi + e^{i\phi} \sin \phi \cdot u}{|\sin \phi| \sqrt{1+u^2}} \quad u = \frac{x + \cos \phi}{\sin \phi} \\
&= \frac{e^{i\phi}}{2\rho} \left(\sqrt{1+2\rho \cos \phi + \rho^2} - \sqrt{1-2\rho \cos \phi + \rho^2} \right. \\
&\quad \left. - i |\sin \phi| \ln \frac{\rho + \cos \phi + \epsilon \sqrt{1+2\rho \cos \phi + \rho^2}}{-\rho + \cos \phi + \epsilon \sqrt{1-2\rho \cos \phi + \rho^2}} \right) \quad \epsilon = \operatorname{sgn} \sin \phi.
\end{aligned} \tag{8}$$

Analogously

$$\begin{aligned}
c_{10} &= \frac{\sqrt{3}}{4\pi} \int_0^{2\pi} d\varphi \cdot \int_{-\rho}^{\rho} \frac{dx}{\rho^2} x \frac{1 + xe^{i\phi}}{\sqrt{1+2x \cos \phi + x^2}} = \\
&= \frac{\sqrt{3}}{8\rho^2} \left(2e^{i\phi} \left((\rho + \cos \phi) \sqrt{1+2\rho \cos \phi + \rho^2} \right. \right. \\
&\quad \left. \left. - (-\rho + \cos \phi) \sqrt{1-2\rho \cos \phi + \rho^2} \right) \right. \\
&\quad \left. - 4e^{2i\phi} \left(\sqrt{1+2\rho \cos \phi + \rho^2} - \sqrt{1-2\rho \cos \phi + \rho^2} \right) \right. \\
&\quad \left. + i(3e^{2i\phi} + 1) |\sin \phi| \ln \frac{\rho + \cos \phi + \epsilon \sqrt{1+2\rho \cos \phi + \rho^2}}{-\rho + \cos \phi + \epsilon \sqrt{1-2\rho \cos \phi + \rho^2}} \right).
\end{aligned} \tag{9}$$

The limits

$$\begin{aligned}
\rho \rightarrow 0 : \quad c_0 &\rightarrow 1, \quad c_{10} \rightarrow 0 && \text{(normal lies in the plane of S-wave)} \\
\rho \rightarrow \infty : \quad c_0 &\rightarrow 0, \quad c_{10} \rightarrow \frac{\sqrt{3}}{2} e^{i\phi} && \text{(normal lies in the plane of P-wave)}
\end{aligned}$$

Fig.3a shows the projection of the region Ω into coordinates $(c_0, |c_{10}|)$. Dotted band is the projection of the boundary $\partial\Omega$, the projection of Ω is shown by sloppy hatching. Fig.3b shows the slices of $\partial\Omega$ by level surfaces $c_0 = \text{Const}$ in projection on Argand plot $(\operatorname{Re} c_{10}, \operatorname{Im} c_{10})$. When $c_0 \rightarrow 1$, the slices contract into a point $(0, 0)$ (the P-wave contribution disappears). When $c_0 \rightarrow 0$, the slice reaches the circle with the radius $\sqrt{3}/2$. For intermediate values $0 < c_0 < 1$ the slices are of the oval shape. In this the length of radius-vector $|c_{10}|$ changes from minimal value to maximal value in the band, displayed on fig.3a. The slices of Ω by surfaces $c_0 = \text{Const}$ lie inside the ovals. Fig.3c shows 3-dimensional region of ambiguity.

Expansions of the functions $\Psi(\vec{n}) = e^{i\varphi(\vec{n})}/\sqrt{4\pi}$ in common contain infinite number of harmonics. Figures display the projections (shades), which Ω casts from infinite dimensional coefficient space in specified 2- or 3-dimensional subspaces. A contribution of the infinite number of harmonics not involved in these figures can be evaluated. We use the equality

$$\sum_0^\infty |c_\alpha|^2 = \int d^2n |\Psi|^2 = 1 \quad \Rightarrow \quad \sum_{\alpha \neq S, P_0} |c_\alpha|^2 = 1 - c_0^2 - |c_{10}|^2.$$

On fig.3a two circles with radii 0.95 and 1 are shown. For the points of Ω , lying between these two circles, the contribution of “invisible” harmonics does not exceed 10% of total probability

$$0.9 < c_0^2 + |c_{10}|^2 \leq 1 \quad \Rightarrow \quad \sum_{\alpha \neq S, P_0} |c_\alpha|^2 < 0.1$$

The same region is hatched on fig.3b. On the bottom figure this region is correspondent to a part of Ω , lying outside a sphere with radius 0.95.

The figures are obtained as follows. 10^6 random points (ρ, ϕ) were generated with uniform distribution in rectangle

$$\{0 < \rho < 30, 0 < \phi < 2\pi\}.$$

Coefficients c_0, c_{10} were calculated by the expressions (8),(9). Redefinition (7) was performed: $c_{10} \rightarrow c_{10}c_0^*/|c_0|$, $c_0 \rightarrow |c_0|$. Values $(c_0, |c_{10}|)$ were placed in 2-dimensional histogram. Besides, 5 distributions were filled on Argand plot ($\text{Re } c_{10}, \text{Im } c_{10}$) for c_0 , lying in the intervals

$$\{p - 0.03 < c_0 < p + 0.03\} \quad p = 0.1, 0.3, 0.5, 0.7, 0.9$$

4) One more symmetry of the region Ω .

In transformations of N_{lm} by matrices $\mathcal{D}_{mm'}^l$ (Wigner’s functions) the coefficients c_{lm} are transformed by the same matrices

$$N_{lm} \rightarrow \sum_{m'} \mathcal{D}_{mm'}^l(\omega) N_{lm'} \Rightarrow c_{lm} \rightarrow \sum_{m'} \mathcal{D}_{mm'}^l(\omega) c_{lm'}$$

(proved by the replacement of integration parameters in (6)).

Unitary matrices $\mathcal{D}_{mm'}^l$ conserve scalar product (3). In $\mathcal{D}_{mm'}^l$ -rotations the region Ω transforms to itself, with its normals.

If one has studied in detail the region Ω in the considered example, then the regions of solutions for non-isotropic distributions $I \neq Const$ can also be obtained. Formula (5) is linear to \sqrt{I} :

$$\text{if } \sqrt{I} = \alpha_1 \sqrt{I_1} + \alpha_2 \sqrt{I_2}, \quad \text{then } c_\beta = \alpha_1 c_\beta^1 + \alpha_2 c_\beta^2.$$

Expanding \sqrt{I} by spherical harmonics: $\sqrt{I} = \sum b_\alpha Y_\alpha$ and using in (5) the multiplication theorem for spherical functions

$$Y_\alpha Y_\beta^* = \text{linear combination of } Y_\beta^*,$$

we see, that the region $\Omega_{I \neq Const}$ can be obtained from the region $\Omega_{I=Const}$ by a linear transformation with coefficients depending on the distribution I .

Substantial questions

1. For the scalar product (3) we have chosen the Euclidean scalar product in real space, formed by real and imaginary parts of c_α . (This scalar product is also equal to real part of the Hermitean scalar product in complex space c_α .) It is clear that the scalar product is an additional structure, which does not influence the shape of the region Ω . One can easily show that the replacement of scalar product leads to some linear transformation of the parameter $N \rightarrow \tilde{N}(N)$ in (5), i.e. the shape of the region conserves, only its parametrization changes. One can imagine this in the following way. The replacement of the scalar product changes the orthogonality definition and, consequently, the definition of the normal to the surface. When the scalar product replaces, the field of normals N on the boundary $\partial\Omega$ changes, but the surface $\partial\Omega$ remains the same.

2. In infinite dimensional space of coefficients c_α the set Ω is not a region. The Ω is a surface, lying on a sphere $\sum_0^\infty |c_\alpha|^2 = \int d^2n I(\vec{n}) = 1$. This surface maps to the region in projections into any less space. Expressions (2) and (5) in initial space coincide ($\Omega = \partial\Omega$), because any function $\varphi(\vec{n})$ (for which $e^{i\varphi(\vec{n})}$ is quadratically integrable) can be presented in the form $\arg \sum_0^\infty N_\alpha Y_\alpha(\vec{n})$. Actually, we consider the projection of Ω in some finite dimensional space from very beginning. In derivation of formula (5) the vector N_α was restricted to this space.

3. In those points of a sphere \vec{n}_0 , where $\sum N_\alpha Y_\alpha(\vec{n}_0) = 0$, the function $\varphi_*(\vec{n}) = \arg \sum N_\alpha Y_\alpha(\vec{n})$ has a break (branching point). The correspondent wave function $\Psi_* = \sqrt{I} e^{i\varphi_*}$ also breaks in this point¹, except a special case $I(\vec{n}_0) = 0$. This circumstance is complicated by the fact, that all functions of form (1), close to Ψ_* , also have breaks. (The phase of Ψ_* changes by 2π in bypassing around \vec{n}_0 . All functions, close to Ψ_* , also possess this property. If $|\Psi_*|^2 = |\Psi|^2 = I \neq 0$ in the vicinity of \vec{n}_0 , then Ψ should break in this vicinity.)

Therefore, Ω includes regions, correspondent to irregular functions. If the regular functions Ψ should be found, for which $|\Psi|^2 = I$ *exactly*, then additional analysis is needed to discard the regions of irregularity. Outside class (1) smooth functions Ψ exist, which are close to Ψ_* everywhere, except a small vicinity of \vec{n}_0 :

$$\forall \varepsilon, \delta \exists \Psi \in C^1(S^2) \quad |\Psi(\vec{n}) - \Psi_*(\vec{n})| < \varepsilon \quad \forall \vec{n} : |\vec{n} - \vec{n}_0| > \delta.$$

Inside δ -vicinity of \vec{n}_0 the difference $\Psi - \Psi_*$ is bounded. (We present functions $\Psi_* = z/|z|$ and $\Psi = z/(|z| + \delta)$ as an example, see fig.4).

The phase of Ψ changes by 2π in bypassing around \vec{n}_0 . From the continuity, a point exists in δ -vicinity of \vec{n}_0 , where the Ψ and the correspondent density $|\Psi|^2$ vanish. Outside δ -vicinity $|\Psi|^2$ can be close to I as one likes. Coefficients c_α for Ψ are close to the coefficients for Ψ_* . Therefore, the distinction of Ψ and Ψ_* is not physically significant.

II. Let us consider a fermion, decaying into 2 particles with spins 0 and $\frac{1}{2}$. The initial state is described by 2-component spinor

$$\Psi(\vec{n}) = \begin{pmatrix} \Psi_1(\vec{n}) \\ \Psi_2(\vec{n}) \end{pmatrix}, \quad \Psi_i(\vec{n}) = \sum_{lm} c_{lm}^i Y_{lm}(\vec{n}).$$

The angular distribution observed is

$$I(\vec{n}) = |\Psi_1(\vec{n})|^2 + |\Psi_2(\vec{n})|^2.$$

The space of states is a direct product of an orbital space of decay particles and a spin space of the final particle with the spin $\frac{1}{2}$.

¹However, it is single-valued, bounded and expansible by spherical harmonics in the vicinity of \vec{n}_0 . A detailed study shows that formula (5) holds true for these functions.

The state with definite orbital moment l and projections m, λ is:

$$|lm; \tfrac{1}{2}\lambda\rangle = Y_{lm}(\vec{n})\chi_\lambda, \quad \chi_{+\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The state with definite total moment J and projection M is:

$$|JM\rangle = \sum_{\substack{l=J\pm\frac{1}{2} \\ \lambda=\pm\frac{1}{2}}} \langle l \ M - \lambda; \tfrac{1}{2}\lambda | JM \rangle \cdot |l \ M - \lambda; \tfrac{1}{2}\lambda\rangle.$$

If the coefficients of expansion of each component of the spinor Ψ by spherical harmonics are known

$$\Psi(\vec{n}) = \sum_{lm\lambda} c_{lm}^\lambda Y_{lm}(\vec{n})\chi_\lambda,$$

it is possible to obtain spin composition of the initial state

$$\tilde{c}_{JM} = \langle JM | \Psi \rangle = \sum_{\substack{l=J\pm\frac{1}{2} \\ \lambda=\pm\frac{1}{2}}} c_{l, M-\lambda}^\lambda \langle l \ M - \lambda; \tfrac{1}{2}\lambda | JM \rangle^*.$$

Let us find $\Psi(\vec{n})$.

$$|\Psi_1(\vec{n})| = \sqrt{I(\vec{n})} \cos \phi(\vec{n}), \quad |\Psi_2(\vec{n})| = \sqrt{I(\vec{n})} \sin \phi(\vec{n}),$$

$\phi(\vec{n})$ is an arbitrary function of angles. This function defines a degree of polarization of the final particle with the spin $\frac{1}{2}$, which is not measured in the experiment.

$$\Psi_1(\vec{n}) = \sqrt{I(\vec{n})} \cos \phi(\vec{n}) e^{i\varphi_1(\vec{n})}, \quad \Psi_2(\vec{n}) = \sqrt{I(\vec{n})} \sin \phi(\vec{n}) e^{i\varphi_2(\vec{n})},$$

$$c_{lm}^i = \int d^2n \ Y_{lm}^*(\vec{n}) \sqrt{I(\vec{n})} e^{i\varphi_i(\vec{n})} \cdot \begin{cases} \cos \phi(\vec{n}), & i = 1 \\ \sin \phi(\vec{n}), & i = 2 \end{cases} \quad (10)$$

One can apply to this formula the analysis described above. The result is the following. The boundary of the region of ambiguity is parametrized by complex numbers N_α^i . The point on the boundary $c_\alpha^i(N)$ is defined by the formula (10), where

$$\varphi_i = \arg \sum_\alpha N_\alpha^i Y_\alpha \quad \text{and} \quad \tan \phi = \frac{|\sum_\alpha N_\alpha^2 Y_\alpha|}{|\sum_\alpha N_\alpha^1 Y_\alpha|}.$$

III. Let the process be described by density matrix. We suppose that only spinor components were mixed, i.e. only the information about polarizations is lost. For example, consider the process: $0 + \frac{1}{2} \rightarrow X \rightarrow 0 + \frac{1}{2}$, 2 plane waves scatter in the initial state, the initial fermion is not polarized, the polarization of final fermion is not measured.

The initial state is

$$|i\lambda_i\rangle = \delta(\vec{k} - k_i\hat{z})\chi_{\lambda_i} \sim \sum_l \sqrt{\frac{2l+1}{4\pi}} Y_{l0}(\vec{n}) \cdot \chi_{\lambda_i}.$$

The orbital part of the wave function $|i\rangle$ is known, the spin part is unknown.

The initial state is described by density matrix:

$$\rho_i = \frac{1}{2} \sum_{\lambda_i=\pm\frac{1}{2}} |i\lambda_i\rangle\langle i\lambda_i|.$$

The intermediate state X depends on λ_i : $|\Psi_{\lambda_i}\rangle = M|i\lambda_i\rangle$, where M is a reaction amplitude. Components of the wave function X and coefficients of their expansions by spherical harmonics represent matrix elements of reaction amplitude

$$\Psi_{\lambda_i\lambda_f}(\vec{n}) = \langle \vec{n}\lambda_f | M | i\lambda_i \rangle, \quad |\vec{n}_0\lambda_f\rangle = \delta^2(\vec{n} - \vec{n}_0)\chi_{\lambda_f},$$

$$c_{lm}^{\lambda_i\lambda_f} = \langle lm; \frac{1}{2}\lambda_f | M | i\lambda_i \rangle, \quad |lm; \frac{1}{2}\lambda_f\rangle = Y_{lm}(\vec{n})\chi_{\lambda_f}.$$

The angular distribution observed is

$$\begin{aligned} I(\vec{n}) &= \sum_{\lambda_f=\pm\frac{1}{2}} \langle \vec{n}\lambda_f | \underbrace{M\rho_i M^+}_{\rho_X} | \vec{n}\lambda_f \rangle = \\ &= \frac{1}{2} \sum_{\substack{\lambda_i=\pm\frac{1}{2} \\ \lambda_f=\pm\frac{1}{2}}} \langle \vec{n}\lambda_f | M | i\lambda_i \rangle \langle i\lambda_i | M^+ | \vec{n}\lambda_f \rangle = \frac{1}{2} \sum_{\substack{\lambda_i=\pm\frac{1}{2} \\ \lambda_f=\pm\frac{1}{2}}} |\Psi_{\lambda_i\lambda_f}(\vec{n})|^2 \end{aligned}$$

$$\begin{aligned} \Psi_{\lambda_i\lambda_f}(\vec{n}) &= \sqrt{2I(\vec{n})} e^{i\varphi_{\lambda_i\lambda_f}(\vec{n})} \cdot \begin{cases} \cos \phi_1(\vec{n}), & \lambda_i = +\frac{1}{2} \\ \sin \phi_1(\vec{n}), & \lambda_i = -\frac{1}{2} \end{cases} \cdot \begin{cases} \cos \phi_2(\vec{n}), & \lambda_f = +\frac{1}{2} \\ \sin \phi_2(\vec{n}), & \lambda_f = -\frac{1}{2} \end{cases} \\ c_{\alpha}^{\lambda_i\lambda_f} &= \int d^2n Y_{\alpha}^*(\vec{n}) \Psi_{\lambda_i\lambda_f}(\vec{n}). \end{aligned}$$

This formula contains 6 arbitrary functions. For the points on the boundary these functions have a form:

$$\begin{aligned}
\varphi_{\lambda_i \lambda_f} &= \arg \sum_{\alpha} N_{\alpha}^{\lambda_i \lambda_f} Y_{\alpha} \\
\tan \phi_1 &= \frac{R^{-+2} + R^{--2} - R^{+-2} - R^{++2} \pm \sqrt{D}}{2(R^{-+}R^{++} + R^{--}R^{+-})} \\
\tan \phi_2 &= \frac{-R^{-+2} + R^{--2} + R^{+-2} - R^{++2} \pm \sqrt{D}}{2(R^{-+}R^{--} + R^{+-}R^{++})} \\
D &= (R^{++2} + R^{+-2} - R^{--2} - R^{-+2})^2 + 4(R^{-+}R^{++} + R^{--}R^{+-})^2, \\
\text{where } R^{\lambda_i \lambda_f} &= \left| \sum_{\alpha} N_{\alpha}^{\lambda_i \lambda_f} Y_{\alpha} \right|.
\end{aligned}$$

Notions.

1) In view of axial symmetry of the problem the distribution I does not depend on azimuthal angle. The amplitude Ψ is also proposed to be axially symmetrical, its expansion contains harmonics Y_{l0} only. Derivation of ambiguity region in the space $\{c_{l0}\}$ is equivalent to sectioning of Ω by this space. The result is described by the same expressions, in which the indices α are restricted to the set $(l0)$.

2) All mentioned above corresponds to the case of inelastic scattering (particles in initial and final states are different). For elastic scattering it is necessary to extract explicitly in S -matrix a contribution of the non-scattered wave

$$S = 1 + iM.$$

The consequence of unitarity of S -matrix is an optical theorem for the amplitude of elastic scattering:

$$\text{Im } \Psi|_{\vec{n}=\hat{z}} \sim \sigma_{\text{tot}}.$$

(If the initial state is fixed, then the optical theorem is the unique consequence of the unitarity of S .) This condition define a hyperplane in the space of coefficients. The slice of the region of ambiguity by this hyperplane is the solution of the problem in the case of elastic scattering.

3) For pure elastic scattering the combined imposition of unitarity and rotational invariance of S -matrix fixes the continuous ambiguity (for example, see [2] §125). With nonelastic channels this does not occur. Rotational

invariance leads to diagonality of S -matrix with respect to quantum numbers of the total moment

$$\langle J'M'n'|S|JMn\rangle = \delta_{JJ'}\delta_{MM'}S_{nn'}^{(JM)},$$

index n includes all other quantum numbers, e.g. the type of a particle. The matrices $S_{nn'}^{(JM)}$ are unitary. This condition provides non-linear relations among partial amplitudes of different reactions. An exact inclusion of the unitarity condition is possible only in the measurement and combined analysis of the whole variety of connected channels.

An extensive literature is devoted to the question of continuous ambiguities in phase analysis (see review in [3], pp.291-308). The reaction amplitude is ambiguously reconstructed from experimentally observable quantities, because the part of information about amplitude is lost in the measurement. Non-observable quantities (phases, polarizations) can be functions of angles, energy, etc., therefore the final expression for the amplitude contains functional arbitrariness. This arbitrariness is significant (if one substitutes fast changing phase $\varphi(\vec{n})$ in formula (5), then the contribution of low harmonics decreases, the same distribution is described by high harmonics). One can diminish the ambiguity only imposing extra restrictions on the form of amplitude, i.e. adopting the particular model. Note, that in the case of the known region of ambiguity the inclusion of extra restrictions reduces to sectioning of the region by the surface of the restriction.

For example, it is known that the initial state Ψ has definite parity. In this case one should consider the region Ω in the space $\{c_{lm}, l \text{ have the given parity}\}$. In this one can use formula (5), restricting l in the indices α and β to the set with the given parity. One can prove this considering the slice of Ω by this space or repeating the reasonings in section I for Ψ with the definite parity and restricting the integrations on semi-spheres.

The state $\Psi(\vec{n}) = \sqrt{I(\vec{n})}e^{i\varphi(\vec{n})}$ includes in general an infinite number of harmonics. Coefficients c_{lm} tend to zero at $l \rightarrow \infty$ (fastly, if $\Psi(\vec{n})$ is sufficiently smooth). In practical partial wave analysis the experimental distribution is fitted by the finite segment of the expansion: $\Psi = \sum_0^{L_{\max}} c_{lm} Y_{lm}$, in this the

finite number of solutions has been obtained [4]. This happens because the segment of the expansion approximates the exact solution, and just for the finite number of values of parameters this approximation is the best (fig.5).

At $L_{\max} \rightarrow \infty$ we turn back in infinite dimensional space, where the number of solutions is infinite. Apparently, when L_{\max} increases, the fit becomes unstable or new solutions appear, which fill the region Ω in the limit $L_{\max} \rightarrow \infty$.

There is much speculation that the consideration of high harmonics is not necessary, because noise exceeds a signal in high harmonics due to finite angular resolution of the equipment and limited statistics. The contributions of high harmonics cannot be precisely measured, but certainly this does not mean that they are equal to zero. We have observed that weak and sharp conditions

$$c_{lm} \rightarrow 0 \quad l \rightarrow \infty \quad \text{and} \quad c_{lm} = 0 \quad l > L_{\max}$$

have principal difference. In the first case the large “ambiguity island” exists². In the second case the finite number of solutions will be found, closest to the boundary of the island. The situations are possible when resonant structures in low harmonics can be also described by correlated contributions of high harmonics. One can avoid this uncertainty, if the whole region of ambiguity will be found in analysis. The general scheme of such analysis is the following:

- Energy axis of the system considered is divided in bins, the angular distributions are filled in each bin. These distributions are approximated by smooth functions, e.g. spherical harmonics:

$$I(\vec{n}) = \sum_{lm} t_{lm}^* Y_{lm}(\vec{n}), \quad t_{lm} = \int d^2n \, I(\vec{n}) Y_{lm}(\vec{n}) = \langle Y_{lm} \rangle_I.$$

- The boundary of the ambiguity region is obtained by formula (5). The integral on a sphere entering this formula can be numerically evaluated.

This integral is not expectation value of some quantity, in its calculation the whole distribution $I(\vec{n})$ must be known. Also, it should not be calculated taking a root $\sqrt{I(\vec{n})}$ from each bin content and summing over bins. Many-dimensional distributions are rarely filled and the errors of such integration

²The region of ambiguity presented on fig.3 fills almost the whole unit sphere.

will be great. The approximation of $I(\vec{n})$ by the smooth function in the previous step is necessary.

A many-dimensional surface obtained can be kept in the form of linear frame. A displaying technique for many-dimensional surfaces presented in this form is being developed nowadays [5].

Note, that our approach does not require the fitting of the experimental distribution. This excludes various problems, such as the derivation of all best approximations, which requires either repeated applications of fitting procedure with different start points or complex algebra for discovering analytic relations between different solutions.

A presence of resonance in some wave is established independently on a model only if the projection of ambiguity region in 3-dimensional space

Argand plot for this wave \times energy axis

is a narrow tube containing only resonant paths (when the energy changes, the absolute value of the wave has a bump, the phase changes by π). In other cases the non-resonant paths are present along with resonant ones. In this case one may only state that the experimental data *do not contradict* the presence of the resonance in the given wave.

A model predicts the path $c(E)$ (or the class of paths). If the ambiguity region is obtained, one can easily check, whether the path is contained in the ambiguity region and also find the area of values of model parameters, when the paths are contained in $\Omega(E)$. Particularly, these parameters might be masses and widths of resonant states.

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Received December 8, 1994.

Figures captions

Fig.1. For the point inside the region Ω , the linear span of all possible variations δc_α coincides with the whole space. For the point on the boundary of Ω all δc_α lie in the tangent plane to the boundary surface and are orthogonal to normal vector N_α .

Fig.2. The region Ω , shade and slice.

Fig.3. The ambiguity region for isotropic distribution in projection on SP_0 -waves (see text for explanations).

Fig.4. Function Ψ_* has a break in the point \vec{n}_0 . All functions, close to Ψ_* , for which $|\Psi|^2 = |\Psi_*|^2$, have breaks in the vicinity of \vec{n}_0 . Continuous functions Ψ exist, which are close to Ψ_* everywhere, except the small vicinity of \vec{n}_0 . For these functions the $|\Psi|^2$ vanishes in some point inside the vicinity.

Fig.5. The restriction of the number of harmonics and model substitutions for the amplitude lead to the fact that the solution is considered in some narrow class of functions. This class and the set of exact solutions Ω in general do not intersect each other. The distance between points in these two classes become minimal in a discrete set of points.

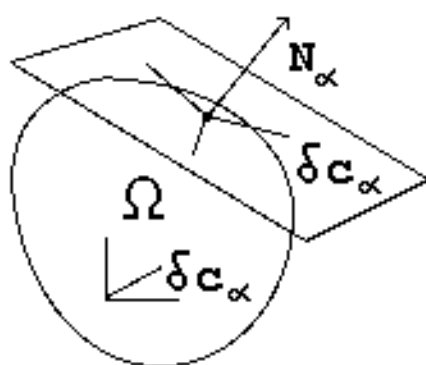


fig.1

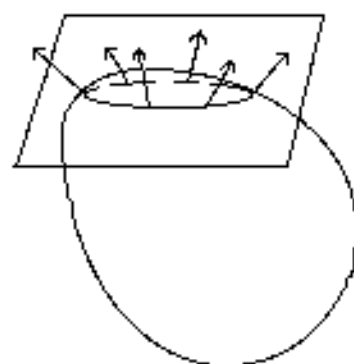
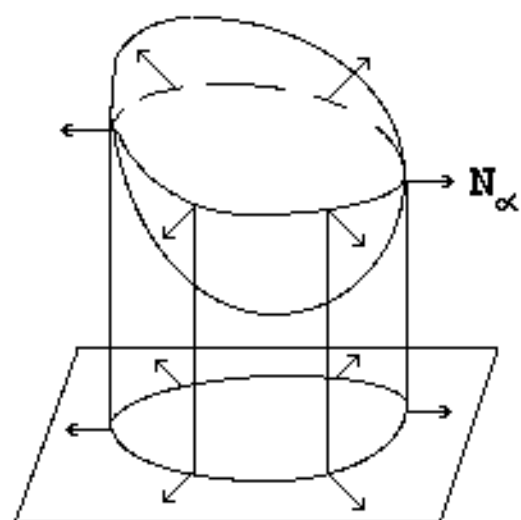


fig. 2

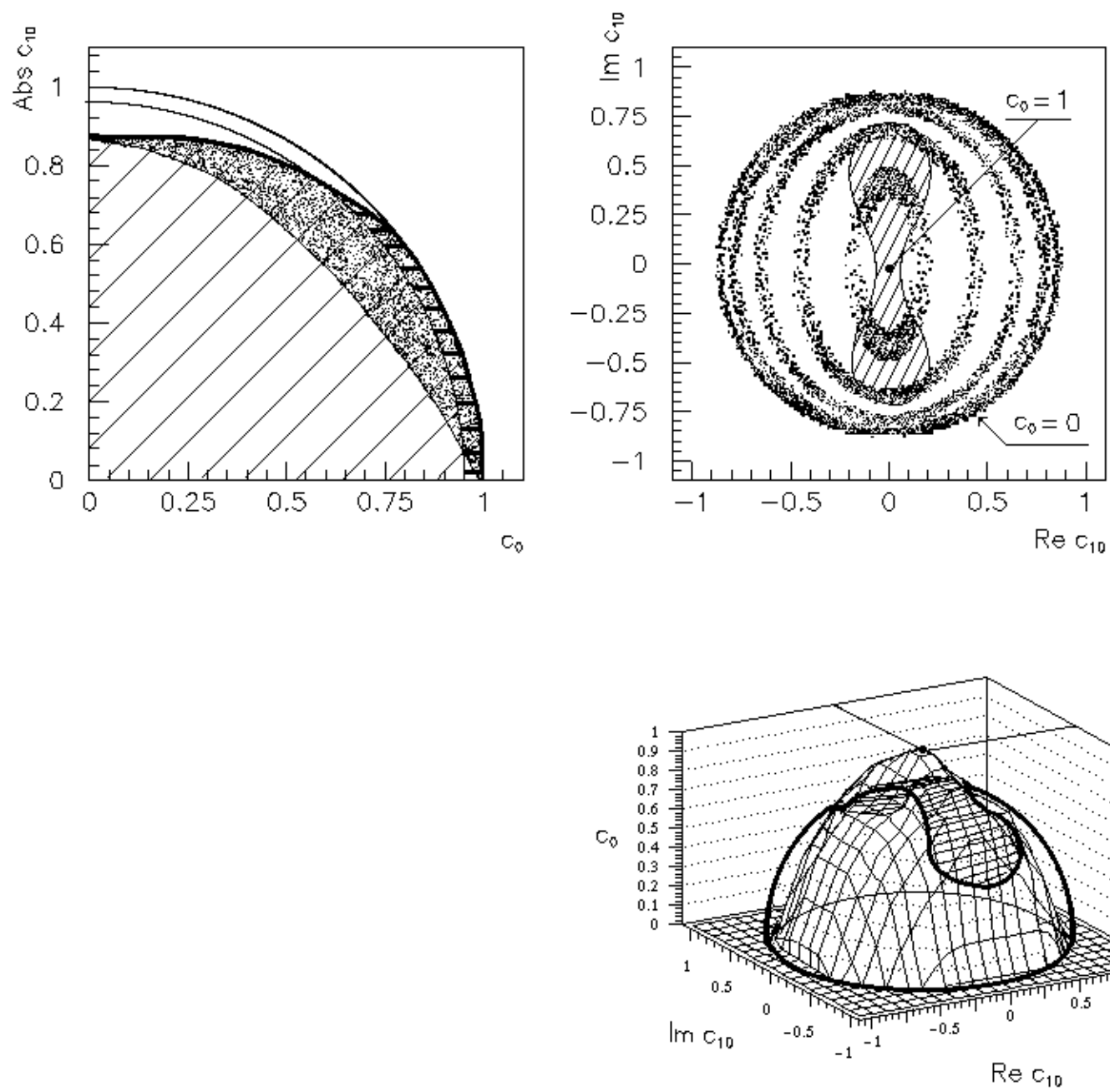


fig. 3

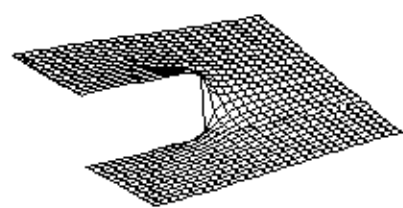
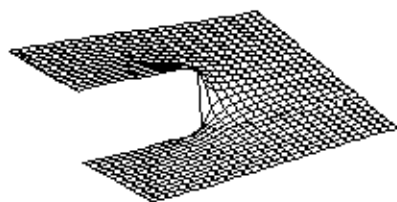
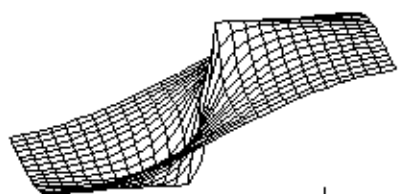
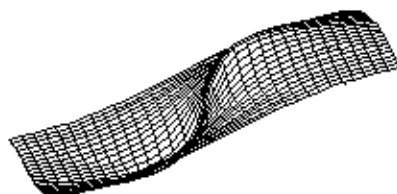
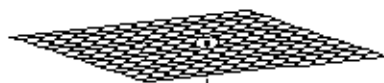
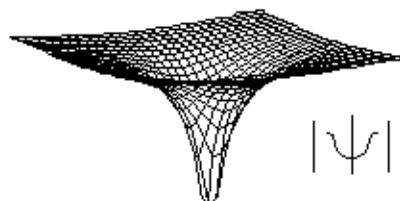

 φ_\star

 φ

 $\text{Re } \psi_\star$

 $\text{Re } \psi$

 $|\psi_\star|^2$
 \vec{n}_0

 $|\psi|^2$

fig. 4

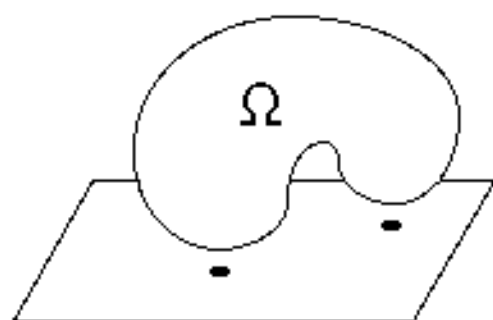


fig. 5